Journal of Functional Analysis **166**, 168–178 (1999)
Article ID jfan.1999.3419, available online at http://www.idealibrary.com on IDEAL®

The Log-Sobolev Inequality for Unbounded Spin Systems

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Received December 23, 1998

In this note, we would like to give a proof of Log-Sobolev inequality for unbounded spin systems with weaker assumptions on the potentials than previously obtained. © 1999 Academic Press

1. INTRODUCTION AND PRELIMINARIES

Our aim is to analyze the thermodynamic properties of the measure $\exp -\Phi^{\Lambda,\omega}(X)\,dX$ in the case when $\Phi^{\Lambda,\omega}$, which is associated with cubes $\Lambda\subset\mathbb{Z}^d$ and some $\omega\in(\mathbb{R})^{\mathbb{Z}^d}$ defining the boundary condition, has the form, for $X=X^\Lambda\in(\mathbb{R})^\Lambda$,

$$\Phi^{\Lambda,\omega}(X)=\sum_{j\in\Lambda}\phi(x_j)+\frac{\mathscr{J}}{2}\sum_{(\{j\}\cup\{k\})\cap\Lambda\neq\varnothing,\, j\sim k}V(z_j-z_k), \tag{1.1}$$

where

- $X=(x_j)_{j\in\Lambda}$,
- ϕ is a one particle phase on $\mathbb{R}$ with at least quadratic increase at ∞,
- V is a convex function on $\mathbb{R}$ with bounded second derivative that is satisfying

$$|V''(t)|\leqslant C, \tag{1.2}$$

0022-1236/99 $30.00

- we let

$$
\begin{aligned}
z_j &= x_j \qquad \text{if} \quad j \in \Lambda \\
z_j &= \omega_j \qquad \text{if} \quad j \notin \Lambda,
\end{aligned} \tag{1.3}
$$

- $j \sim k$ means that j and k are nearest neighbors for the ℓ^1-distance in $\mathbb{Z}^d$,
- we let

$$\mathscr{J} \geqslant 0. \tag{1.4}$$

We shall sometimes use the decomposition

$$\Phi^{\Lambda,\omega} = \Phi_d^{\Lambda} + \mathscr{J}\Phi_i^{\Lambda,\omega}, \tag{1.5}$$

with

$$\Phi_d^{\Lambda}(X) = \sum_{j \in \Lambda} \phi(x_j), \tag{1.6}$$

and

$$\Phi_i^{\Lambda,\omega}(X) = \sum_{(\{j\} \cup \{k\}) \cap \Lambda \neq \varnothing,\, j \sim k} V(z_j - z_k). \tag{1.7}$$

Our main assumption is an assumption of convexity at ∞ of the single spin phase ϕ. We assume that there exists a bounded C^∞ function s such that $\tilde{\phi} := \phi + s$ is strictly convex. More precisely, there exists $\rho > 0$ such that

$$(\phi + s)''(t) \geqslant \rho > 0, \qquad \forall t \in \mathbb{R}. \tag{1.8}$$

The typical example (which was the main case treated in [Yo1]) is

$$\phi(x) = \tfrac{1}{12}\lambda x^4 + \tfrac{1}{2}\nu x^2, \tag{1.9}$$

where the parameters λ and ν satisfy

$$\lambda > 0, \tag{1.10}$$

and ν may be negative.

B. Zegarlinski gave in [Ze] a sufficient condition for uniform logarithmic inequalities. Later N. Yoshida proposed in [Yo1] an alternative proof, also using ideas of Lu and Yau [LuYa], but his proof is based on stronger assumptions on the single spin phase: assumptions (U1) and (U2) say in particular that the single spin phase is superquadratic and that the phase $(x, y) \mapsto \theta(x, y) := \phi(x+y) + \phi(x-y)$ obtained by duplication has a special property: $xy(\partial^2/\partial x\, \partial y)\theta \geqslant 0$.

Both proofs use a method consisting in duplicating the measure and working on a conditional measure depending on new integration variables. This had the drawback of needing a clustering property for a conditional measure stronger than the one actually needed. The technique, we develop in this paper requires only the uniform decay of correlations for the Gibbs measure (which is the key tool in the proof of Log-Sobolev) and not the stronger condition introduced in [Ze, Yo1].

Unlike the papers [Ze, Yo1], we are mainly interested in the perturbative regime. This enables us to generalize the hypotheses on the potentials: the proof is valid even for quadratic potentials and for very general two-bodies interaction. The main ingredient is the uniform decay of correlations proven for example in [He3]. Nevertheless we follow closely the scheme of the proof of [Yo1] (see also [LuYa]), thus we only focus on the key lemmas which needed to be modified.

Our main problem will be to analyze the properties of the Gibbs measure

$$dE^{\Lambda,\omega} := \exp -\Phi^{\Lambda,\omega}(X)\, dX \Big/ \left(\int_{(\mathbb{R})^\Lambda} \exp -\Phi^{\Lambda,\omega}(X)\, dX \right), \tag{1.11}$$

and in particular the existence of Log-Sobolev inequalities attached to this family of probability measures.

For this, we analyze the covariance associated to $f, g \in C^\infty_{temp}((\mathbb{R})^\Lambda)$

$$E^{\Lambda,\omega}(f; g) := \mathrm{Cov}_{\Lambda,\omega}(f, g) = \langle (f - \langle f \rangle_{\Lambda,\omega})(g - \langle g \rangle_{\Lambda,\omega}) \rangle_{\Lambda,\omega}, \tag{1.12}$$

where $\langle \cdot \rangle_{\Lambda,\omega}$ denotes the mean value with respect to the measure $dE^{\Lambda,\omega}$ and $C^\infty_{temp}((\mathbb{R})^\Lambda)$ is the space of C^∞ functions with polynomial growth.

We recall that it has been proven in [He3] the following theorem.

THEOREM 1.1. *Under the previous assumptions, there exists $\mathscr{J}_0 > 0$, $c > 0$, and $\lambda > 0$ such that the following inequality holds for all functions f and g in $C^\infty_{temp}((\mathbb{R})^\Lambda)$*

$$|E^{\Lambda,\omega}(f; g)| \leqslant \lambda \exp(-cd(S_f, S_g))\, E^{\Lambda,\omega}(|\nabla f|^2)^{1/2}\, E^{\Lambda,\omega}(|\nabla g|^2)^{1/2}, \tag{1.13}$$

uniformly with respect to the other parameters $\Lambda \subset \mathbb{Z}^d$, $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, $\mathscr{J} \in [0, \mathscr{J}_0]$.

This inequality with $f = g$ implies that the spectral gap is greater than some constant $1/\lambda$ uniformly with respect to the same parameters.

2. SOME LOG-SOBOLEV INEQUALITY FOR EFFECTIVE SINGLE SPIN PHASE

If we follow Yoshida's approach the main lemma we shall need is

LEMMA 2.1. *Let us assume that* ϕ *satisfies* (1.8), *then there exists* $\mathcal{J}_0$ *positive such that, for any* $n \in \mathbb{N}$, *there exists a constant* C_n *such that, for any* $\mathcal{J} \in [0, \mathcal{J}_0]$, *any* Δ, Λ *contained in* $\mathbb{Z}^d$, *any* $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, *any* f *such that* $S_f \cap \Lambda \subset \Delta$ *and* $|\Delta| \leqslant n$, *we have*

$$E^{\Lambda, \omega}\left(f^2 \ln \frac{f^2}{E^{\Lambda, \omega}(f^2)}\right) \leqslant C_n E^{\Lambda, \omega}(|\nabla_\Lambda f|^2). \tag{2.1}$$

Here $E^{\Lambda, \omega}$ *is the expectation with respect to the measure* $(1/Z_{\Lambda, \omega}) \exp -\Phi^{\Lambda, \omega}\, dX^\Lambda$.

When $\Lambda = \Delta = \{i\}$, this is a Log-Sobolev inequality for a single spin effective phase $\phi_j(t) = \phi(t) + \mathcal{J} \sum_{k \sim j} V(t - \omega_k)$. Under the assumption (1.8), this inequality is a direct consequence of Bakry–Emery argument applied to $\tilde{\phi}_j(t) = \tilde{\phi}(t) + \mathcal{J} \sum_{k \sim j} V(t - \omega_k)$. The conclusion of the lemma seems much stronger than this uniform inequality for the family of single spin effective phases ϕ_j.

Proof of the Lemma. For $\Delta \subset \Lambda$ we define the probability measure $E_\Delta^{\Lambda, \omega}$ on $\mathbb{R}^\Delta$ as the projection of $E^{\Lambda, \omega}$

$$E_\Delta^{\Lambda, \omega}(f) := E^{\Lambda, \omega}(f \otimes 1_{\Lambda \setminus \Delta}). \tag{2.2}$$

What we need is finally a uniform Log-Sobolev inequality with a constant depending only of the cardinal of Δ: $|\Delta|$.

In order to take the same notations as in [Yo1], we introduce

$$J_{ij} = \mathcal{J} \qquad \text{if} \quad i \sim j, \qquad J_{ij} = 0 \qquad \text{else}. \tag{2.3}$$

The phase appearing in the density of this measure with respect to dX^Δ has the form

$$\Phi_\Delta^{\Lambda, \omega}(X^\Delta) = \Phi^{\Delta, f}(X^\Delta) + \Psi_\Delta^{\Lambda, \omega, \mathcal{J}}(X^\Delta),$$

where $\Phi^{\Delta, f}$ is the phase inside Δ

$$\Phi^{\Delta, f}(X^\Delta) = \sum_{j \in \Delta} \phi(x_j) + \sum_{j, k \in \Delta} J_{jk} V(x_j - x_k).$$

Changing by a multiplicative constant $\exp 2(\sup |s|)\, |\Delta|$, it is enough to treat the case of a measure where ϕ is replaced by $\tilde{\phi}$ and consequently uniformly strictly convex on $\mathbb{R}$ (see [DeSt2, Corollary 6.2.45]).

For this we have just to control the Hessian of $\Psi_\Delta^{\Lambda, \omega, \mathscr{J}}(X^\Delta)$ with respect to the variables X^Δ.

Here we have (up to an irrelevant multiplicative constant)

$$\exp - \psi_\Delta^{\Lambda, \omega, \mathscr{J}}(X^\Delta)$$
$$:= \int (\exp - \Phi^{\Lambda \setminus \Delta, f}(X^{\Lambda \setminus \Delta}))(\exp - \Phi^{\Delta, \Lambda \setminus \Delta, \omega}(X^{\Lambda \setminus \Delta}))\, dX^{\Lambda \setminus \Delta}, \quad (2.4)$$

with

$$\Phi^{\Lambda \setminus \Delta, f}(X^{\Lambda \setminus \Delta}) := \sum_{k \in \Lambda \setminus \Delta} \phi(x_k) + \sum_{i,\, j \in \Lambda \setminus \Delta} J_{ij} V(x_i - x_j), \quad (2.5)$$

and

$$\Phi^{\Delta, \Lambda \setminus \Delta, \omega}(X^{\Lambda \setminus \Delta}) := \sum_{i \in \Lambda \setminus \Delta,\, j \in \Delta \cup \Lambda^c} J_{ij} V(x_i - z_j). \quad (2.6)$$

Here $z_j = x_j$ if $j \in \Delta$ and ω_j if $j \in \Lambda^c$. We observe also that $\Phi^{\Delta, f}$ is uniformly strictly convex when ϕ is replaced by $\tilde{\phi}$ (also independently of $\mathscr{J} \geqslant 0$). We treat the second term as a perturbation for $\mathscr{J}$ small enough. What is relevant here is the Hessian of $\Psi_\Delta^{\Lambda, \omega, \mathscr{J}}(X^\Delta)$. When computing this Hessian we get, using the convexity of V, the following comparison between $(|\Delta| \times |\Delta|)$ matrices

$$(\text{Hess}\, \Psi)_{(i,\, j)} \geqslant - \left[\sum_{k,\, \ell \in \Lambda \setminus \Delta} J_{ik} J_{j\ell} \operatorname{Cov}_{\Lambda \setminus \Delta,\, z}(V'(x_k - \omega_i), V'(x_\ell - \omega_j)) \right]_{(i,\, j)}, \quad (2.7)$$

where the variables $\omega_i = x_i$ and $\omega_j = x_j$ are fixed because the covariance takes only into account the spins with indices in $\Lambda \setminus \Delta$.

In order to estimate this Hessian, one has to use the uniform decay of the correlations for $E^{\Lambda \setminus \Delta, z}$ (see Theorem 1.1) and we get

$$\forall i, j \in \Delta, \qquad |\operatorname{Cov}_{\Lambda \setminus \Delta,\, z}(V'(x_k - \omega_i), V'(x_\ell - \omega_j))| \leqslant C \exp - \frac{1}{C} d(k, \ell), \quad (2.8)$$

since V'' is bounded. One is reduced to the following upper bounds:

$$\sum_{i,\, j \in \Delta} |\xi_i|\, |\xi_j| \sum_{k,\, \ell \in \Lambda \setminus \Delta} |J_{ik}|\, |J_{j\ell}| \exp - \frac{1}{C} d(k, \ell)$$
$$\leqslant \sum_{k,\, \ell \in \Lambda \setminus \Delta} \exp - \frac{1}{C} d(k, \ell) \left(\sum_{i \in \Delta} |J_{ik}|\, |\xi_i| \right) \left(\sum_{j \in \Delta} |J_{j\ell}|\, |\xi_\ell| \right)$$
$$\leqslant \text{const.}\ \|\mathbf{J}\|^2\, \|\xi\|^2.$$

Here we have defined $\|\mathbf{J}\|$ by

$$\|\mathbf{J}\| := \sup_j \sum_k |J_{jk}| = 2\mathscr{J} d.$$

For $\mathscr{J}$ small enough, this cannot perturb the strict convexity of our phase and one can apply the Bakry–Emery argument [BaEm] to the phase $\Psi_{\Delta}^{\Lambda,\omega,\mathscr{J}}(X^{\Delta})$ (with $\tilde{\phi}$ strictly convex) to get the Logarithmic Sobolev inequality. We have completed the proof of the lemma.

Remark 2.2. Let us suppose that the conclusion of Theorem 1.1 holds for some $\mathscr{J} > 0$; then inspection of the proof shows that, if the single spin phase is super-quadratic, the conclusion of Lemma 2.1 is true for this $\mathscr{J} > 0$. This is the case studied by Yoshida.

3. THE ROLE OF THE DECAY ESTIMATES FOR THE LOG-SOBOLEV INEQUALITY

These decay estimates were already present in the proof of the previous lemma (but not in a decisive way). They were introduced in the context of the Log-Sobolev inequalities by B. Zegarlinski. N. Yoshida replaced them by a stronger mixing condition, leading for technical reasons to restrictions on the single spin-phase.[1] We show here that this is actually not necessary because we do not need to use the auxiliary variables (p, q) introduced by N. Yoshida.

The proof of the Log-Sobolev inequality by N. Yoshida rests on a second lemma (Lemma 3.4 of [Yo1]) which says:

LEMMA 3.1. *Let us assume that* (1.13) *and the conclusions of Lemma* 2.1 *are satisfied. Then there exists* $\mathscr{J}_0$, C *and* c *such that, for any* $\mathscr{J} \in [0, \mathscr{J}_0]$, $\Lambda \subset \mathbb{Z}^d$, $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, $i \notin \Lambda$, *and any* f *sufficiently regular*

$$|\nabla_i \sqrt{E^{\Lambda,\omega}(f^2)}| \leqslant \sqrt{E^{\Lambda,\omega}(|\nabla_i f|^2)} + C\,|\Delta| \exp(-cd(i, S_f))$$
$$\times \left(E^{\Lambda,\omega}(|\nabla_\Delta f|^2) + E^{\Lambda,\omega}\left(f^2 \log\left(\frac{f^2}{E^{\Lambda,\omega}(f^2)}\right)\right)\right)^{1/2},$$

where $\Delta = S_f \cap \Lambda$ *and* S_f *is the support of* f.

[1] In [Yo1], some superquadratic increase is in particular assumed (see assumptions (U1) and (U2)). For polynomials ϕ: $\phi(t) = \sum_{\nu=1}^{m} a_\nu t^{2\nu}$, N. Yoshida imposes, for example, the conditions $m > 1$, $a_m > 0$, and $a_\nu \geqslant 0$ for $\nu > 1$.

Proof. Following Yoshida, we compute

$$|\nabla_i \sqrt{E^{\Lambda,\omega}(f^2)}| \leqslant \tfrac{1}{2} E^{\Lambda,\omega}(f^2)^{-1/2} \times \left(2\,|E^{\Lambda,\omega}(f\nabla_i f)| + \sum_{j\in\Lambda,\, j\sim i} |E^{\Lambda,\omega}(f^2;\, V'(\omega_i - x_j))|\right). \tag{3.1}$$

The variable $\omega_i = x_i$ is fixed. In the following we keep this notation to remind the reader that x_i belongs to the boundary conditions. This leads to

$$|\nabla_i \sqrt{E^{\Lambda,\omega}(f^2)}| \leqslant \sqrt{E^{\Lambda,\omega}(|\nabla_i f|^2)} + \tfrac{1}{2} E^{\Lambda,\omega}(f^2)^{-1/2} \sum_{j\in\Lambda,\, j\sim i} |E^{\Lambda,\omega}(f^2;\, V'(\omega_i - x_j))|.$$

First one applies the Dobrushin–Lanford–Ruelle (DLR) equations and takes the conditional expectation with respect to S_f. More concretely, this means that we can write, with $j \in \Lambda$ and $j \notin \Delta$

$$E^{\Lambda,\omega}(f^2;\, V'(\omega_i - x_j)) = E^{\Lambda,\omega}(f^2;\, E^{\Lambda\setminus S_f,\, z}(V'(\omega_i - x_j))), \tag{3.2}$$

where $z_\ell = x_\ell$ if $\ell \in S_f$ and $z_\ell = \omega_\ell$ if $\ell \notin \Lambda$.

The covariance can be rewritten, using a duplication[2] of the variables, as

$$\begin{aligned} E^{\Lambda,\omega}&(f^2;\, V'(\omega_i - x_j)) \\ &= (E^{\Lambda,\omega} \times \tilde{E}^{\Lambda,\omega})((f^2 - \tilde{f}^2)(E^{\Lambda\setminus S_f,\, z}(V'(\omega_i - x_j)) \\ &\quad - E^{\Lambda\setminus S_j,\, \tilde{z}}(V'(\omega_i - x_j)))) \end{aligned} \tag{3.3}$$

where $\tilde{z}_\ell = \tilde{x}_\ell$ if $\ell \in S_f$ and $\tilde{z}_\ell = \omega_\ell$ if $\ell \notin \Lambda$.

Using Taylor's formula, we get

$$E^{\Lambda\setminus S_f,\, z}(V'(\omega_i - x_j)) - E^{\Lambda\setminus S_f,\, \tilde{z}}(V'(\omega_i - x_j)) = \int_0^1 h'(t)\, dt, \tag{3.4}$$

with $z_t = (1-t)\, z + t\tilde{z}$ and $h(t) := E^{\Lambda\setminus S_f,\, z_t}(V'(\omega_i - x_j))$.

[2] If μ is a probability measure on Ω, we can always write the covariance of two functions u and v in the form $\mathrm{Cov}_\mu(u, v) = \int_{\Omega\times\Omega} (u(x) - u(\tilde{x})) \cdot (v(x) - v(\tilde{x}))\, d\mu(x) \cdot d\mu(\tilde{x})$.

Hence we have

$$\begin{aligned}|E^{\Lambda\backslash S_f, z}(V'(\omega_i-x_j)) - E^{\Lambda\backslash S_f, \tilde z}(V'(\omega_i-x_j))| \\ \leqslant \sup_{\theta;\, n\in\delta(S_f);\, \ell\in S_f;\, \ell\sim n} |E^{\Lambda\backslash S_f, \theta}(V'(\theta_\ell - x_n);\, V'(\omega_i-x_j))| \\ \times\left(\sum_{k\in S_f} |z_k-\tilde z_k|\right).\end{aligned} \tag{3.5}$$

Here $\delta(S_f)=\{m\in\Lambda\backslash S_f \mid d(m, S_f)=1\}$.

The uniform decay of correlations for the Gibbs measure and the boundedness of V'' imply by Theorem 1.1

$$\begin{aligned}&\forall i\notin\Lambda,\quad \forall j\in\Lambda \text{ s.t. } i\sim j,\\ &|E^{\Lambda,\omega}(f^2;\, V'(\omega_i-x_j))|\\ &\qquad\leqslant \lambda\exp(-cd(i, S_f))\sum_{k\in S_f}(E^{\Lambda,\omega}\times\tilde E^{\Lambda,\omega})(|f^2-\tilde f^2|\,|x_k-\tilde x_k|).\end{aligned} \tag{3.6}$$

Therefore, it remains to prove that there is a constant C such that, for any k in Λ,

$$\begin{aligned}&(E^{\Lambda,\omega}\times\tilde E^{\Lambda,\omega})(|f^2-\tilde f^2|\,|x_k-\tilde x_k|)\\ &\quad\leqslant CE^{\Lambda,\omega}(f^2)^{1/2}\left(E^{\Lambda,\omega}(|\nabla f|^2)+E^{\Lambda,\omega}\left(f^2\log\left(\frac{f^2}{E^{\Lambda,\omega}(f^2)}\right)\right)\right)^{1/2}.\end{aligned} \tag{3.7}$$

First we apply Cauchy–Schwarz inequality

$$\begin{aligned}&(E^{\Lambda,\omega}\times\tilde E^{\Lambda,\omega})(|f^2-\tilde f^2|\,|x_k-\tilde x_k|)\\ &\quad\leqslant((E^{\Lambda,\omega}\times\tilde E^{\Lambda,\omega})((f-\tilde f)^2))^{1/2}\\ &\qquad\times((E^{\Lambda,\omega}\times\tilde E^{\Lambda,\omega})((f+\tilde f)^2\,(x_k-\tilde x_k)^2))^{1/2}.\end{aligned} \tag{3.8}$$

Using the spectral gap estimate of Theorem 1.1 (see [He3] and also [Yo1] for a different approach)

$$(E^{\Lambda,\omega}\times\tilde E^{\Lambda,\omega})((f-\tilde f)^2)\leqslant\min\{4E^{\Lambda,\omega}(f^2),\, E^{\Lambda,\omega}(|\nabla f|^2)\}. \tag{3.9}$$

In order to bound the other term, we first observe that

$$(E^{\Lambda,\omega}\times\tilde E^{\Lambda,\omega})((f+\tilde f)^2\,(x_k-\tilde x_k)^2)\leqslant 4(E^{\Lambda,\omega}\times\tilde E^{\Lambda,\omega})(f^2(x_k-\tilde x_k)^2). \tag{3.10}$$

We then use the following entropic inequality (see Deuschel and Stroock [DeSt2, p. 68])

$$\forall t>0,\qquad \mu(uv)\leqslant\frac{1}{t}\log(\mu(\exp(tv)))+\frac{1}{t}\mu(u\log u), \tag{3.11}$$

where μ is any probability measure, v a function and the function u is a density ($u \geqslant 0$ and $\mu(u) = 1$). Applying this inequality, we get for $k \in \Lambda$,

$$
\begin{aligned}
&(E^{\Lambda,\omega} \times \tilde{E}^{\Lambda,\omega})(f^2(x_k - \tilde{x}_k)^2) \\
&\quad \leqslant \frac{1}{t} E^{\Lambda,\omega}(f^2) \log((E^{\Lambda,\omega} \times \tilde{E}^{\Lambda,\omega})(\exp t(x_k - \tilde{x}_k)^2)) \\
&\quad + \frac{1}{t} E^{\Lambda,\omega}\left(f^2 \log\left(\frac{f^2}{E^{\Lambda,\omega}(f^2)}\right)\right).
\end{aligned} \tag{3.12}
$$

One first notes that, by symmetry of the measure, we have

$$
(E^{\Lambda,\omega} \times \tilde{E}^{\Lambda,\omega})(x_k - \tilde{x}_k) = 0. \tag{3.13}
$$

Furthermore, one knows from Lemma 2.1 that the Log-Sobolev inequality holds for the measure

$$
d\mu = Z_{\Lambda,\omega}^{-1} \exp -\Phi^{\Lambda,\omega}\, dX^{\Lambda}
$$

and for functions depending only of the variable x_k with a constant which is uniform with respect to ω and this is also the case for the duplicate measure when restricted to function depending on x_k and $\tilde{x}_k$.

We follow Ledoux's proof of Herbst's argument for the duplicated measure $d\mu := dE_{\Lambda,\omega}(X^{\Lambda}) \otimes dE_{\Lambda,\omega}(\tilde{X}^{\Lambda})$ with functions depending only on two variables x_k and $\tilde{x}_k$. One knows that for t sufficiently small, there is $c > 0$ such that uniformly in ω,

$$
(E^{\Lambda,\omega} \times \tilde{E}^{\Lambda,\omega})(\exp t(x_k - \tilde{x}_k)^2) \leqslant c. \tag{3.14}
$$

Here we have also used the property (3.13). Therefore there exists $c > 0$ and $t > 0$ such that

$$
\begin{aligned}
&(E^{\Lambda,\omega} \times \tilde{E}^{\Lambda,\omega})(f^2(x_k - \tilde{x}_k)^2) \\
&\quad \leqslant \frac{c}{t} E^{\Lambda,\omega}(f^2) + \frac{1}{t} E^{\Lambda,\omega}\left(f^2 \log\left(\frac{f^2}{E^{\Lambda,\omega}(f^2)}\right)\right).
\end{aligned} \tag{3.15}
$$

Combining the previous inequalities, one derives (3.7).

4. END OF THE PROOF OF THE LOG-SOBOLEV INEQUALITY

Here we refer for the end of the proof to the presentation of Yoshida [Yo1]. Roughly speaking, N. Yoshida's proof of this second part is essentially the transposition to the continuous case of a proof established in some discrete case by Lu and Yau [LuYa].

Hence we have finally obtained

THEOREM 4.1. *Let us assume that* (1.8) *is satisfied. There exists* $\mathcal{J}_0 > 0$ *such that for* $\mathcal{J} \in [0, \mathcal{J}_0]$, *there are constants* $C, \kappa \in [0, +\infty[$ *such that, for any* $\Lambda \subset \mathbb{Z}^d$ *and any* $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, *we have*

$$|\mathrm{Cov}_{\Lambda, \omega}(x_i, x_j)| \leqslant C \exp -\kappa\, d(i, j), \qquad \forall i, j \in \Lambda. \tag{4.1}$$

Furthermore there exists a constant $c \in]0, +\infty[$ *such that for any* $\Lambda \subset \mathbb{Z}^d$ *and any* $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, *we have*

$$\langle f \ln f \rangle_{\Lambda, \omega} \leqslant 2c \langle |\nabla f^{1/2}|^2 \rangle_\Lambda + \langle f \rangle_{\Lambda, \omega} \ln \langle f \rangle_{\Lambda, \omega}, \tag{4.2}$$

for all nonnegative functions f for which the right-hand side is finite.

This theorem was previously obtained in [He3] as a consequence of Zegarlinski's theorem.

Remark 4.2. If we analyze what we have done, we have indeed obtained some variant of Zegarlinski's theorem in the following form: properties (1.13) and (2.1) imply uniform Log-Sobolev inequalities.

As we recalled before, the property (1.13) was obtained in [He3], for $\mathcal{J} \geqslant 0$ small enough, under the assumption (1.8) and in fact under weaker conditions. The proof of (2.1) under assumption (1.8) is what we have done in the first part of this note. Let us recall that (1.8) was also assumed in Zegarlisnki's Theorem.

ACKNOWLEDGMENTS

We thank N. Yoshida for sending us preliminary versions of his papers prior to publication. We are grateful to B. Zegarlinski and J. D. Deuschel for very enlightening discussions. This work was supported by the European Union through the TMR network—Network Post-doctoral training program (FMRX-CT 960001) in partial differential equations and applications in quantum mechanics.

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Printed in Belgium